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Product of Partition Logics, Orthoalgebras, and Automata

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We attempt to define a coupled system consisting of two partition logics and we introduce a product of partition logics. These partition logics have a close connection with Moore and Mealy-type automata. We show how the coupled system of two automata is connected with the product of partition logics, and present some illustrative examples.

1. INTRODUCTION

In the axiomatic approach to quantum mechanics, the event structure of a physical system is identified with a quantum logic (Busch *et al.*, 1991) or an orthoalgebra (Foulis *et al.*, 1992; Randall and Foulis, 1981), while in the case of classical mechanics, it is identified with a Boolean algebra (Pták and Pulmannová, 1991). Assume that we have two independent physical systems with event structures P and Q and we wish to regard them as a coupled system. The event structure L of this coupled system is usually called a tensor product of P and Q and we write $L = P \otimes Q$.

Tensor products in various approaches have been studied in Aerts and Daubechies (1978), Foulis (1989), Foulis and Bennett (1993), Foulis and Pták (1995), Foulis and Randall (1981), Kläy *et al.* (1987), Lock (1990), Matolcsi (1975), Randall and Foulis (1981), Wilce (1990), and Zecca (1978). A tensor product of orthoalgebras has been investigated by Foulis and Bennett (1993) via a universal mapping property, and a tensor product of an orthoalgebra and a Boolean algebra is given in Foulis and Pták (1995).

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Somewhat more general structures than orthoaglebras are quasi-orthoalgebras, for which a binary operation $\oplus$ is not, in general, associative. Very important examples of quasi-orthoalgebras, partition logics, have an intimate connection (Dvurečenskij *et al.*, 1995) with the investigation of the empirical propositional structure of Moore and Mealy-type automata.

The aim of the present paper is to introduce a weaker form of a tensor product of partition logics, a product of partition logics, and to show its connection with automata. We show that this structure exists.

2. ORTHOALGEBRAS

The notion of orthoalgebras (or quasi-orthoalgebras) goes back to axiomatic models of quantum mechanics introduced by Foulis and Randall (1981; Randall and Foulis, 1981) as special algebraic structures describing propositional logics.

A *quasi-orthoalgebra* is a set L endowed with two special elements 0, $1 \in L$ $(0 \neq 1)$ and equipped with a partially defined binary operation $\oplus$ satisfying the following conditions for all $a, b \in L$:

(oa-i) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$ (commutativity law).

(oa-ii) $a \oplus 0$ is defined for any $a \in L$ and $a \oplus 0 = a$.

(oa-iii) For any $a \in L$, there is an element $a' \in L$ such that $a \oplus a'$ is defined and $a \oplus a' = 1$ (orthocomplementation law).

(oa-iv) If $a \oplus (a' \oplus b)$ is defined, then $b = 0$.

(oa-v) If $a \oplus (a \oplus b)$ is defined, then $a = 0$.

(oa-vi) If $a \oplus b$ is defined, then $a \oplus (a \oplus b)'$ is defined and $b' = a \oplus (a \oplus b)'$.

The following facts are true:

Proposition 2.1. Let L be a quasi-orthoalgebra, $a, b \in L$. Then:
(a) $0' = 1$, $1' = 0$.
(b) $(a')' = a$.
(c) If $a \oplus b = a \oplus c$, then $b = c$.
(d) If $a \oplus b = 1$, then $b = a'$.

The unique element a' is called the *orthocomplement* of $a \in L$, and the unary operation $'$: $L \to L$ defined by $a \mapsto a'$, $a \in L$, is said to be an *orthocomplementation*. We shall say that two elements $a, b \in L$ (i) are *orthogonal*, and write $a \perp b$, iff $a \oplus b$ is defined in L (it is clear that $a \perp b$ iff $b \perp a$), and (ii) $a \leq b$ iff there is an element $c \in L$ with $a \oplus c = b$.

It is easily shown that the relation $\leq$ is reflexive and antisymmetric, but need not be transitive. An associative quasi-orthoalgebra, i.e., a quasi-orthoalgebra, for which the associative law

(oa-vii) If $a \oplus b$, $(a \oplus b) \oplus c$ are defined in L, so are $b \oplus c$ and $a \oplus (b \oplus c)$, and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

holds is said to be an *orthoalgebra* (OA). If in L there are the elements $(a \oplus b) \oplus c$ and $a \oplus b(\oplus c)$ and if they coincide, we denote it as $a \oplus b \oplus c$. In any orthoalgebra, $\leq$ is transitive. On other hand, it is possible to give an example of a quasi-orthoalgebra with transitive $\leq$ which does not correspond to any orthoalgebra. If the elements $a \oplus (b \oplus c)$ and $(a \oplus b) \oplus c$ exist in a quasi-orthoalgebra and if they coincide, we denote them as $a \oplus b \oplus c$.

Due to Golfin (1987), an orthoalgebra is a set L with two special elements $0, 1 \in L$ $(0 \neq 1)$ and endowed with a partial binary operation $\oplus$ satisfying (oa-i), (oa-ii), (oa-iii), (oa-vii), and:

(oa-v*) If $a \oplus a$ is defined, then $a = 0$.

The original idea of the partial binary operation $\oplus$ goes back to Boole's pioneering work (Boole, 1854), where he wrote $a + b$ as the logical disjunction of events a and b when the logical conjunction $ab = 0$, so that, for mutually excluding events a and b, $a + b$ is defined. This is all that is needed for probability theory: if $ab = 0$, then $P(a + b) = P(a) + P(b)$. To avoid confusion, we write $a \oplus b$ for $a + b$ when $ab = 0$.

Note that one can rewrite axioms for a Boolean algebra in terms of Boole's ideas of $a + b$. For more details, see Foulis and Bennett (1993).

In addition, let L be an orthomodular poset (OMP) (or an orthomodular lattice, OML), i.e., a poset L with the least and last elements 0 and 1 and a unary operation $^\perp$: $L \to L$, called an *orthocomplementation*, such that, for all $a, b \in L$:

(i) $(a^\perp)^\perp = a$.
(ii) If $a \leq b$, then $b^\perp \leq a^\perp$.
(iii) $a \vee a^\perp = 1$.
(iv) If $a \leq b^\perp$ (and we write $a \perp b$), then $a \vee b \in L$.
(v) If $a \leq b$, then $b = a \vee (a \vee b^\perp)^\perp$.

(For OML, L has to be additionally a lattice.) Then L can be organized into an OA if the binary operation $\oplus$ is defined via $a \oplus b$ exists in L iff $a \leq b^\perp$ and $a \oplus b := a \vee b$. The unary operation $'$: $L \to L$ is defined via $a' := a^\perp$, $a \in L$.

We recall that if L is an OA and $a, b \in L$ are mutually orthogonal, then $a, b \leq a \oplus b$, and $a \oplus b$ is the minimal upper bound for a and b (i.e., $a, b \leq a \oplus b$, and if there is $c \in L$ with $a, b \leq c \leq a \oplus b$, then $c = a \oplus b$), but this does not mean that $a \vee b$ exists in L, so that L cannot necessarily be an OMP.

A subset A of a quasi-OA (OA) L is a *quasi-suborthoalgebra* (*suborthoalgebra*) of L if (i) $0, 1 \in A$; (ii) if $a \in A$, then $a' \in A$; (iii) $a, b \in A$ with $a \perp b$ implies $a \oplus b \in L$.

If a (quasi) suborthoalgebra A of L is, in addition, a Boolean algebra with respect to $\leq$, A is called a *Boolean suborthoalgebra* of L. Denote by $\vee_A$ and $\wedge_A$ the join and the meet taken only in A, respectively. Then, $a \oplus b = a \vee_A b$ whenever $a, b \in A$ and L is an OA. A maximal Boolean suborthoalgebra of L is called a *block*.

3. PARTITION LOGICS

In this section, we present a notion of partition logics which will have an intimate connection with special types of automata, and which will generalize the results of Svozil (1993) and Schaller and Svozil (1994, 1995, 1996).

Let L be a quasi-orthoalgebra with $\leq$. A nonvoid subset I of L is said to be an *ideal* of L if:

(i) If $a \in I$, $b \in L$, $b \leq a$, then $b \in I$.
(ii) $a, b \in I$ with $a \perp b$ implies $a \oplus b \in I$.

It is clear that $0 \in I$. An ideal I of L is said to be (i) *proper* if $I \neq L$ or, equivalently, $1 \notin I$; (ii) *prime* if, for any $a \in L$, either $a \in I$ or $a' \in I$. We denote by $P(L)$ the set of all prime ideals in L.

A *probability measure* (or also a *state*) on L is a mapping s: $L \to [0, 1]$ such that (i) $s(1) = 1$ and (ii) $s(a \oplus b) = s(a) + s(b)$ whenever $a \perp b$. A probability measure s is *two-valued* if $s(a) \in \{0, 1\}$ for any $a \in L$.

We recall that there is a one-to-one correspondence between two-valued probability measures and prime ideals: If s is a two-valued probability measure, then $I_s = \{a \in L: s(a) = 0\}$ is a prime ideal; and if I is a prime ideal, then s_I: $L \to [0, 1]$ defined via $s_I(a) = 0$ iff $a \in I$, otherwise $s_I(a) = 1$, is a two-valued probability measure on L.

A set $\mathcal{S}$ of probability measures on L is called *separating* if for all $a, b \in L$, $a \neq b$, there is a probability measure $s \in \mathcal{S}$ such that $s(a) \neq s(b)$. L is called *prime* iff it has a separating set of two-valued probability measure or, equivalently, for any different elements $a, b \in L$ there is a prime ideal I of L such that $a \in I$ and $b \notin I$.

Let $\mathcal{L}$ be a family of quasi-orthoalgebras (or OAs, OMPs, Boolean algebras, etc.) satisfying the following conditions: For all $P, Q \in \mathcal{L}$, $P \cap Q$ is a quasi-suborthoalgebra (subOA, sub OMP, Boolean subalgebra, etc.) of both P and Q, and the partial orderings and orthocomplementations coincide on $P \cap Q$. Define the set $L = \cup := \cup\{P: P \in \mathcal{L}\}$, a relation $\oplus$, and the unary operation $'$ as follows:

(i) $a \oplus b$ iff there is a $P \in \mathcal{L}$ such that $a, b \in P$ and $a \perp_P b$, then $a \oplus b = a \oplus_P b$.

(ii) $a' = b$ iff there is a $P \in \mathcal{P}$ such that $a, b \in P$ and $a'^P = b$.

The set L with the above-defined $\oplus$ is called the *pasting* of the family $\mathcal{L}$.

Let $\mathcal{R}$ be a family of finite partitions of a fixed set X. The pasting of the family of Boolean algebras $\{B_R: R \in \mathcal{R}\}$ is called a *partition logic*, and we denote it as a couple $(X, \mathcal{R})$.

We note that, for $a, b \in L = (X, \mathcal{R})$, $a \oplus b$ is defined on L iff there exists a decomposition $R \in \mathcal{R}$ such that $a, b \in B(R)$ and $a \cap b = \emptyset$, where $B(R)$ is a Boolean algebra generated by R; then $a \oplus b := a \cup b$.

We recall that two quasi-orthoalgebras L_1 and L_2 are *isomorphic* iff there is a one-to-one mapping $\phi: L_1 \to L_2$ such that $a \oplus b$ is defined in L_1 iff $\phi(a) \oplus \phi(b)$ is defined in L_2 and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$.

The following result (Dvurečenskij *et al.*, 1995) describes quasi-orthoalgebras isomorphic to partition logics.

Theorem 3.1. A quasi-orthoalgebra L is isomorphic to a partition logic if and only if L is prime.

It is worth noting that if in a prime quasi-orthoalgebra L there exist elements $x := (a \oplus b) \oplus c$ and $y := a \oplus (b \oplus c)$, for $a, b, c \in L$, then $x = y$. Indeed, by Theorem 3.1, there is a separating system of two-valued probability measures on L, $\mathcal{S}$, and a probability measure $s \in \mathcal{S}$ such that $s(x) \neq s(y)$. Then $s(a \oplus b) + s(c) \neq s(a) + s(b \oplus c)$, which gives the contradiction $s(a) + s(b) + s(c) \neq s(a) + s(b) + s(c)$.

At any rate, the existence of one of x or y in L does not imply the existence of the second one in L; see Example 4.1 below.

4. COUPLED SYSTEMS OF PARTITION LOGICS

The tensor product of orthoalgebras in the category of orthoalgebras was studied by Foulis and Bennett (1993). They showed that if both orthoalgebras P and Q have "enough" probability measures, then the tensor product of P and Q exists. However, they found an example of an orthoalgebra, the Fano plane (Fig. 1), for which the tensor product $P \otimes P$ fails in the category of orthoalgebras. Dvurečenskij (1995) showed that if we use a more general structure, effect algebras, then the tensor product of the Fano plane with itself exists in the category of effect algebras. In general, if both effect algebras P and Q have a nonempty system of probability measures, then the tensor product $P \otimes Q$ exists in the category of effect algebras.

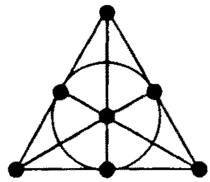

Fig. 1. The Fano plane.

Partition logics have a separating system of two-valued probability measures; unfortunately, they are not, in general, effect algebras, because they are not even (associative) orthoalgebras. Therefore, for the tensor product of partition logics we have to look for another way to introduce it.

4.1. Product of Partition Logics

Assume that $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ are two partition logics, where $\mathcal{R}$ and $\mathcal{S}$ are two systems of finite decompositions of X and Y, respectively. Choose $R \in \mathcal{R}$ and $S \in \mathcal{S}$. Then $R \times S := \{A \times B: A \in R, B \in S\}$ is a decomposition of $X \times Y$. Define

$$\mathcal{R} \times \mathcal{S} := \{R \times S: R \in \mathcal{R}, S \in \mathcal{S}\}$$

Then the partition logic $\{X \times Y, \mathcal{R} \times \mathcal{S}\}$ is called the *product partition logic* of $(X, \mathcal{R})$ and $(Y, \mathcal{S})$.

Example 4.1. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ and let B_1 and B_2 be the Boolean algebras generated by $R_1 := \{\{1\}, \{2\}, \{3\}, \{4\}, \{5, 6\}\}$ and $R_2 := \{\{1\}, \{2\}, \{3, 4\}, \{5\}, \{6\}\}$, respectively. The system $(\Omega, \{R_1, R_2\})$ is a partition logic which is not a Boolean algebra, it is only a quasi-orthoalgebra. Then $(\Omega \times \Omega, \{R_1 \times R_1, R_1 \times R_2, R_2 \times R_1, R_2 \times R_2\})$ is a product partition logic of $(\Omega, \{R_1, R_2\})$ with itself.

Proposition 4.2. Let μ and ν be probability measures on the partition logics $(X, \mathcal{R})$ and $(Y, \mathcal{S})$, respectively, where $\text{card}(Y) = n$ and $\mathcal{S} = \{\{y_1\}, \ldots, \{y_n\}\}$, $y_i \in Y$ for $i = 1, \ldots, n$, $y_i \neq y_j$ for $i \neq j$, and $n \geq 1$. Then there is a unique probability measure $\mu \times \nu$ on $(X \times Y, \mathcal{R} \times \mathcal{S})$ such that

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B), \qquad A \in (X, \mathcal{R}), \quad B \in (Y, \mathcal{S}) \tag{1}$$

If, in addition, μ and ν are two-valued measures, so is $\mu \times \nu$.

Proof. Let U be any element of the product partition logic $L = (X \times Y, \mathcal{R} \times \mathcal{S})$. Without loss of generality we can suppose that $Y = \{1, \ldots, n\}$. It is easy to show that U can be represented uniquely in the form $U = \bigcup_{i=1}^{n} A_i \times \{i\}$, where all A_i belong to the same Boolean algebra B_R generated by a decomposition $R \in \mathcal{R}$; the case $A_i = \emptyset$ is not excluded.

In addition, $U := \cup_{i=1}^{n} A_i \times \{i\} \perp \cup_{i=1}^{n} B_i \times \{i\} =: V$ iff $\{A_i\}_{i=1}^{n}$ and $\{B_i\}_{i=1}^{n}$ belong to the same Boolean algebra B_R and $A_i \cap B_i = \emptyset$ for $i = 1, \ldots, n$, and in this case

$$\bigcup_{i=1}^{n} A_i \times \{i\} \oplus \bigcup_{i=1}^{n} B_i \times \{i\} = \bigcup_{i=1}^{n} (A_i \cup B_i) \times \{i\} = \bigcup_{i=1}^{n} (A_i \oplus B_i) \times \{i\}$$

If now we define $(\mu \times \nu)(U) := \Sigma_{i=1}^{n} \mu(A_i)\nu(\{i\})$, then $\mu \times \nu$ is a well-defined mapping on L. It is simple to verify that $X \times Y = \cup_{i=1}^{n} X \times \{i\}$. Hence

$$(\mu \times \nu)(1) = (\mu \times \nu)(X \times Y) = \sum_{i=1}^{n} \mu(X)\nu(\{i\}) = \mu(X)\nu(Y) = 1$$

Similarly,

$$\begin{aligned}(\mu \times \nu)(U \oplus V) &= (\mu \times \nu)\left(\bigcup_{i=1}^{n} (A_i \oplus B_i) \times \{i\}\right) \\ &= \sum_{i=1}^{n} \mu(A_i \oplus B_i)\nu(\{i\}) \\ &= \sum_{i=1}^{n} (\mu(A_i) + \mu(B_i))\nu(\{i\}) \\ &= (\mu \times \nu)(U) + (\mu \times \nu)(V)\end{aligned}$$

so that $\mu \times \nu$ is a probability measure on L satisfying (1). The uniqueness of $\mu \times \nu$ is now evident. ∎

The probability measure $\mu \times \nu$, if it exists, is said to be a *product probability measure* of μ and ν. We note that the extension of $\mu \times \nu$ from (1) to a probability measure on the product of general partition logics seems to be open.

4.2. Tensor Product of Prime Quasi-Orthoalgebras

According to Theorem 3.1, any prime quasi-orthoalgebra is isomorphic to a partition logic. Therefore, the coupled system consisting of two prime orthoalgebras P and Q being again a prime quasi-orthoalgebra can be called a tensor product of P and Q and it is denoted as $P \otimes Q$. In addition, they have an intimate connection with the product of isomorphic partition logics. For a rigorous introduction of the tensor product we need the following definitions.

Let P and L be two quasi-orthoalgebras. A mapping $\phi: P \to L$ is said to be:

(i) A *morphism* iff $\phi(1) = 1$, and $p \perp q$, $p, q \in P$, implies $\phi(p) \perp \phi(q)$ and $\phi(p \oplus q) = \phi(p) \oplus \phi(q)$.

(ii) A *monomorphism* iff ϕ is a morphism and $\phi(p) \perp \phi(q)$ iff $p \perp q$.

(iii) An *isomorphism* iff ϕ is a surjective monomorphism.

Let P, Q, L be quasi-orthoalgebras. A mapping $\beta: P \times Q \to L$ is called a *bimorphism* iff:

(i) $a, b \in P$ with $a \perp b$, $q \in Q$ imply $\beta(a, q) \perp \beta(b, q)$ and $\beta(a \oplus b, q) = \beta(a, q) \oplus \beta(b, q)$.

(ii) $c, d \in Q$ with $c \perp d$, $p \in P$ imply $\beta(p, c) \perp \beta(p, d)$ and $\beta(p, c \oplus d) = \beta(p, c) \oplus \beta(p, d)$.

(iii) $\beta(1, 1) = 1$.

If $\beta: P \times Q \to L$ is a bimorphism, then $\beta(\cdot, 1): P \to L$ and $\beta(1, \cdot): Q \to L$ are morphisms. Therefore, for $p \in P$ and $q \in Q$, we have $\beta(p, 1)^{\perp} = \beta(p^{\perp}, 1)$, $\beta(1, q)^{\perp} = \beta(1, q^{\perp})$, and $\beta(p, 0) = \beta(0, q) = 0$.

Also, if $a, b, p \in P$ and $c, d, q \in Q$, we have $a \leq b \Rightarrow \beta(a, q) \leq \beta(b, q)$ and $c \leq d \Rightarrow \beta(p, c) \leq \beta(p, d)$.

Definition 4.3. Let P and Q be prime quasi-orthoalgebras. We say that a pair (T, τ) consisting of a prime quasi-orthoalgebra T and a bimorphism $\tau: P \times Q \to T$ is a tensor product of P and Q iff the following conditions are satisfied:

(i) If L is a quasi-orthoalgebra and $\beta: P \times Q \to L$ is a bimorphism, there exists a morphism $\phi: T \to L$ such that $\beta = \phi \circ \tau$.

(ii) For every element of $t \in T$, there is a block B of T such that $t = \bigoplus_{i=1}^{n} \tau(p_i, q_i)$, where $\tau(p_i, q_i)$ belongs to the block B for $i = 1, \ldots, n$.

It is not hard to show that if a tensor product (T, τ) of P and Q exists, it is unique up to an isomorphism, i.e., if (T, τ) and (T^*, τ^*) are tensor products of D-posets P and Q, then there is a unique isomorphism $\phi: T \to T^*$ such that $\phi(\tau(p, q)) = \tau^*(p, q)$ for all $p \in P$, $q \in Q$. Unless confusion threatens, we usually refer to $P \otimes Q$ rather than to $(P \otimes Q, \otimes)$ as being a tensor product.

Unfortunately, we do not know the conditions under which the tensor product of prime quasi-orthoalgebras exists in the category of prime quasi-orthoalgebras.

If both components P and Q are prime orthoalgebras, then the tensor product of P and Q exists in the category of orthoalgebras. This follows from Theorem 6.1 of Foulis and Bennett (1993). Indeed, if $p \in P$ and $q \in Q$ are two nonzero elements of the prime orthoalgebras P and Q, then there exist two probability measures μ and ν on P and Q such that $\mu(p) = 1 = \nu(q)$ [if $p = 1$ then $\mu(1) = 1$ for any μ; if $p \neq 1$, then $0 \neq p^{\perp} \neq 1$ and the primeness and the separateness of P entail the existence of a two-valued probability measure μ with $0 = \mu(p^{\perp}) \neq \mu(1) = 1$]. By Theorem 6.1 of Foulis and Bennett (1993), this is a sufficient condition for the existence of the tensor product of P and Q in the category of orthoalgebras. In addition, for all probability measures μ and ν on P and Q, respectively, the product measure $\mu \times \nu$, defined by $\mu \times \nu(p \otimes q) = \mu(p)\nu(q)$, $p \in P$, $q \in Q$, exists. However, we do not know whether this tensor product also exists in the category of prime quasi-orthoalgebras.

It seems more hopeful to consider the coupled system consisting of a prime quasi-orthoalgebra and a Boolean algebra, called a bounded Boolean power of a prime quasi-orthoalgebra; see also similar problems in Dvurečenskij (1995) for D-posets and Foulis and Pták (1995) for orthoalgebras. We hope to present some results for these coupled systems in the future.

5. REALIZATION BY AUTOMATA

The product of finite automata logics has an intuitive and rather simple realization: If the sets of states of two automata $M_1 = (S_1, I_1, O_1, \delta_1, \lambda_1)$ and $M_2 = (S_2, I_2, O_2, \delta_2, \lambda_2)$ are mutually disjunct, then the automaton partition logic of the *parallel decomposition* $M_1 \| M_2$ of the two automata is the product $\{S_1 \times S_2, \mathcal{R}_1 \times \mathcal{R}_2\}$ of the automaton partition logics $\{S_1, \mathcal{R}_1\}$ and $\{S_2, \mathcal{R}_2\}$ associated with M_1 and M_2, respectively. For a definition of the notation, see below.

5.1. Moore and Mealy Automata

A *finite sequential machine* or *automaton* is a device with the following properties (Hartmanis and Stearns, 1966; Hopcroft and Ullman, 1979; Moore, 1956): (i) a finite set of inputs which can be applied in a sequential order; (ii) a finite set of internal configurations or states; (iii) a finite set of outputs; (iv) a setup such that the present internal configuration and input uniquely determine the next internal configuration and the output.

A *Moore* (*Mealy*) *automaton* is a quintuple $M = (S, I, O, \delta, \lambda)$, where:

(i) S is a finite (nonempty) set of states.
(ii) I is a finite (nonempty) set of inputs.
(iii) O is a finite (nonempty) set of outputs.
(iv) δ: $S \times I \to S$ is a computable transition function.
(v) λ: $S \to O$ is a computable output function (Moore automaton).
(v′) λ: $S \times O \to O$ is a computable output function (Mealy automaton).

In what follows and if not mentioned otherwise, s, i, and o stand for a particular internal state, input, and output, respectively. Moore (Mealy) machines are represented by flow tables and state graphs.

Example 5.1. To illustrate this, assume a Mealy machine $M_s = (S, I, O, \delta, \lambda)$ which has n states, n inputs, and two outputs. That is,

$$S = \{1, 2, \ldots, n\}$$
$$I = \{1, 2, \ldots, n\}$$
$$O = \{0, 1\}$$

Its transition and output functions are ($\delta_{s,x}$ stands for the Kronecker delta function)

$$\delta(s, i) = i$$
$$\lambda(s, i) = \delta_{s,i} = \begin{cases} 1 & \text{if} \quad s = i \\ 0 & \text{if} \quad s \neq i \end{cases}$$

The flow table and state graph of this automaton are given in Fig. 2.

$M_s =$

s/i	1	2	⋯	n	1	2	⋯	n
1	1	2	⋯	n	1	0	⋯	0
2	1	2	⋯	n	0	1	⋯	0
⋮	1	2	⋯	n	0	0	⋯	0
n	1	2	⋯	n	0	0	⋯	1

(a)

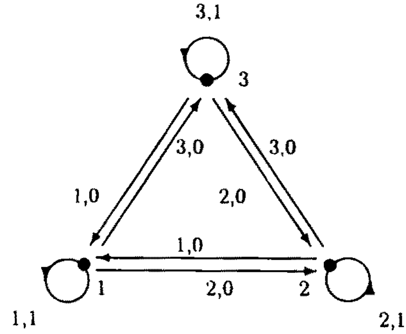

Fig. 2. The flow table and state graph of the Mealy automaton of Example 5.1.

5.2. Machine Isomorphism, Serial and Parallel Decompositions

Two automata $M_1 = (S_1, I_1, O_1, \delta_1, \lambda_1)$ and $M_2 = (S_2, I_2, O_2, \delta_2, \lambda_2)$ of the same type are *isomorphic* if and only if there exist three one-to-one mappings $f\colon S_1 \to S_2$, $g\colon I_1 \to I_2$, and $h\colon O_1 \to O_2$ such that $f[\delta_1(s_1, i_1)] = \delta_2[f(s_1), g(i_1)]$ and $f[\lambda_1(s_1, i_1)] = \lambda_2[f(s_1), g(i_1)]$, where $s_j \in S_j$ and $i_j \in I_j$, $j \in \{1, 2\}$. The triple (f, g, h) is an *isomorphism* between M_1 and M_2. An isomorphism just renames the states, the inputs, and the outputs. From a purely input/output point of view, g as well as h (or h^{-1}) are combinatory circuits and M_1 performs similarly to the serial decomposition (see below) $h^{-1}M_2 g$ of the machines g, M_2, and h^1.

The *serial connection* of the two machines $M_1 = (S_1, I_1, O_1, \delta_1, \lambda_1)$ and $M_2 = (S_2, I_2, O_2, \delta_2, \lambda_2)$ for which $O_1 = I_2$ is the machine in Hartmanis and Stearns (1966, p. 42)

$$M = M_1 \to M_2 = (S_1 \times S_2, I_1, O_2, \delta, \lambda)$$

where $\delta[(s_1, s_2), i] = (\delta_1(s_1, i), \delta_2[s_2, \lambda(s_1, i)])$ and $\lambda[(s_1, s_2), i] = \lambda_2[s_2, \lambda_1(s, i)]$.

The *parallel connection* of the two machines $M_1 = (S_1, I_1, O_1, \delta_1, \lambda_1)$ and $M_2 = (S_2, I_2, O_2, \delta_2, \lambda_2)$ is the machine in Hartmanis and Stearns (1966, p. 48)

$$M = M_1 \| M_2 = (S_1 \times S_2, I_1 \times I_2, O_1 \times O_2, \delta, \lambda)$$

where $\delta[(s_1, s_2), (i_1, i_2)] = (\delta_1(s_1, i_1), \delta_2(s_2, i_2))$ and $\lambda[(s_1, s_2), (i_1, i_2)] = (\lambda_1(s_1, i_1), \lambda_2(s_2, i_2))$.

The logical structure of the initial-state identification problem can be defined as follows. Let us call a proposition concerning the initial state of the machine *experimentally decidable* if there is an experiment E which determines the truth value of that proposition. This can be done by performing E, i.e., by the input of a sequence of input symbols $i_1, i_2, i_3, \ldots, i_n$ associated with E, and by observing the output sequence

$$\lambda_E(s) = \lambda(s, i_1), \lambda(\delta(q, i_1), i_2), \ldots, \lambda(\underbrace{\delta(\cdots\delta(q, i_1)\cdots, i_{n-1}), i_n}_{n-1 \text{ times}})$$

The most general form of a prediction concerning the initial state s of the machine is that the initial state s is contained in a subset P of the state set S. Therefore, we may identify propositions concerning the initial state with subsets of S. A subset P of S is then identified with the proposition that the initial state is contained in P.

Let E be an experiment (a preset or adaptive one), and let $\lambda_E(s)$ denote the obtained output of an initial state s. λ_E defines a mapping of S to the set

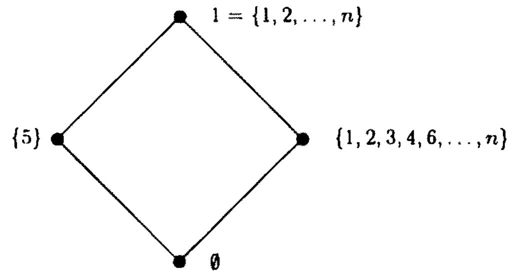

Fig. 3. Haase diagram for Example 5.2.

of output sequences O^*. We define an equivalence relation on the state set S by

$$s \overset{E}{\equiv} t \quad \text{iff} \quad \lambda_E(s) = \lambda_E(t)$$

for any $s, t \in S$. We denote the partition of S corresponding to $\overset{E}{\equiv}$ by $S/\overset{E}{\equiv}$. Obviously, the propositions decidable by the experiment E are the elements of the Boolean algebra generated by $S/\overset{E}{\equiv}$, denoted by B_E.

Let $\mathfrak{R}$ be the set of all Boolean algebras B_E. We call the partition logic $(S, \mathfrak{R})$ an *automaton propositional calculus*.

Example 5.2. In what follows, we explicitly construct the Mealy automaton M_s introduced before. Input/output experiments can be performed by the input of one symbol i (in this example, more inputs yield no finer partitions). Let us assume that one input $i = 5$. This experiment is able to distinguish between state $s = 5$ and all the other states; hence it induces a partition (suppose $n > 5$)

$$v(5) = \{\{5\}, \{1, 2, 3, 4, 6, \ldots, n\}\}$$

After this experiment, information about the initial state is lost (irreversible model). Now consider the partitions $v(i)$ of all possible experiments with one input x (all of them noncomeasurable). Every one of them generates a Boolean algebra of events with two atoms; e.g., $v(5)$ generates a two-element Boolean algebra 2^2 whose Hasse diagram is drawn in Fig. 3.

The *automaton propositional calculus* and the associated *partition logic* are the set of all partitions

$$P = \{v(i) \mid i \in I\}$$

Lattice-theoretically, this amounts to a pasting (Navara and Rogalewicz, 1991) of all the $v(i)$'s. In the specific example, the pasting is just the horizontal sum—only the least and greatest elements $\emptyset$ and $\{1, 2, \ldots, n\}$ of each 2^2 is identified—and one obtains a Chinese lantern lattice MO_n.

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